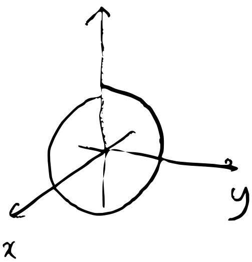
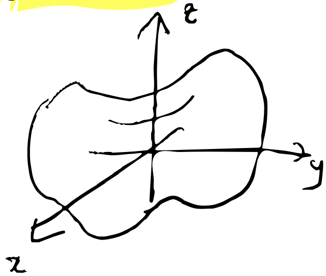
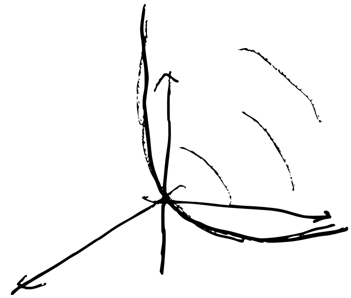
What is a surface? It is a two-dimensional object. Note that to talk about interesting 2-d objects, we have to work in 3D.





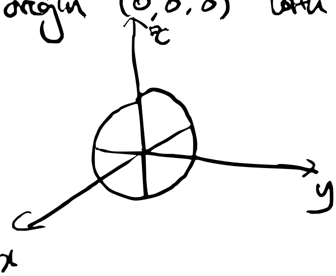


Since we want to express 2-dimensional objects in an equation, it is hard to use parametric equations, because then you would need two parameters. (particles can move in two different directions).

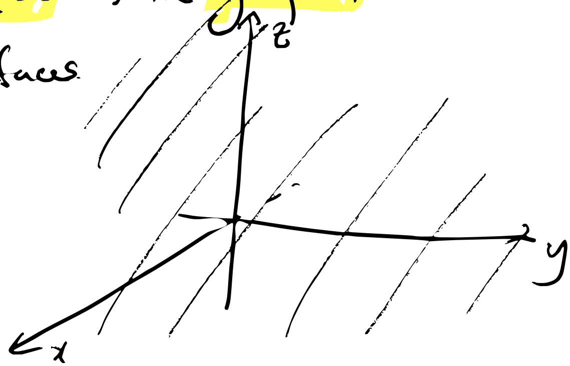
So you usually only work with implicit equations.

Examples

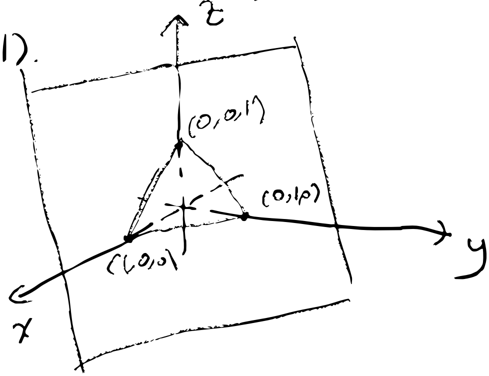
- $x^2+y^2+z^2=1$ is the equation for the sphere centered at the origin $(0,0,0)$ with radius 1.



- $x=0$ is the yz-plane, which is also an example of surfaces.

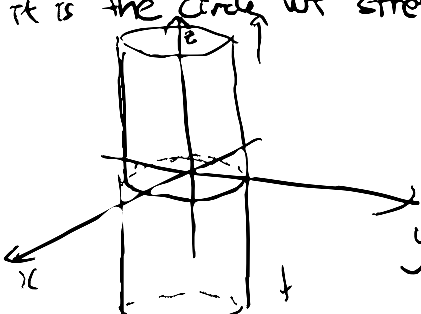


- $x+y+z=1$ is also a plane that contains $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



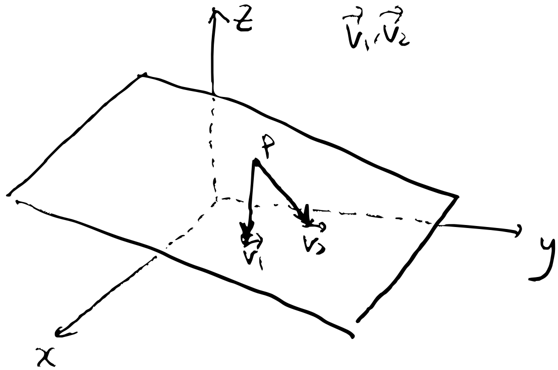
- $x^2+y^2=1$ in 3D is a surface! Unlike 2D, in 3D we have extra variable, z, and z can be anything. So it is the circle but stretched out indefinitely in z-direction.

This is called a cylinder.



A plane is characterized by specifying

- a point that lies on it, P
- and two vectors that are parallel to it. $\vec{v}_1, \vec{v}_2$

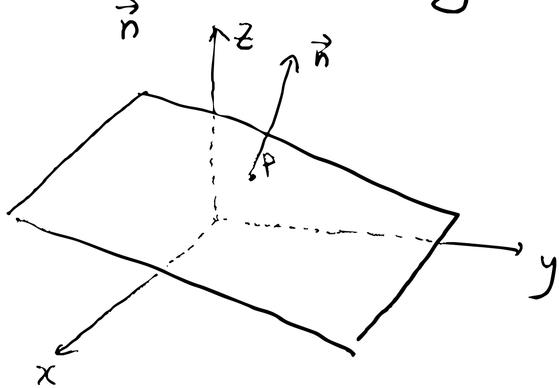


Then a point Q is on this plane if

$$\overrightarrow{PQ} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$ for some c_1, c_2 (scalars).

But this can be more neatly packaged into the following:

- a point P that lies on it,
- and a vector that is orthogonal to the plane. $\vec{n}$



Then a point Q is on the plane if

$\vec{PQ}$ is orthogonal to $\vec{n}$, or $\vec{PQ} \cdot \vec{n} = 0$.

Such orthogonal vector is called a normal vector

If $P = (a_1, a_2, a_3)$ and $\vec{n} = \langle n_1, n_2, n_3 \rangle$, then

$Q = (x, y, z)$ is on this plane if

$\vec{PQ} \cdot \langle n_1, n_2, n_3 \rangle = 0$, or

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0.$$

Any implicit equation of form

$ax + by + cz = d$ represents a plane, and

in particular $\langle a, b, c \rangle$ is a normal vector.

<u>Example</u> An equation of the plane that passes through $P = (1, 3, 5)$ and has normal vector $\vec{n} = \langle -1, 0, 4 \rangle$ can be obtained as

$$(-1) \cdot (x-1) + 0 \cdot (y-3) + 4 \cdot (z-5) = 0, \quad \text{or}$$

$$-x + 1 + 4z - 20 = 0, \quad \text{or} \quad -x + 4z - 19 = 0.$$

How do we transform

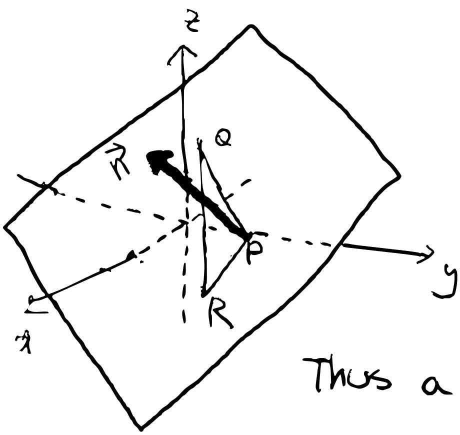
point P + 2 parallel vectors $\vec{v_1}, \vec{v_2}$ $\rightsquigarrow$ point P + normal vector $\vec{n}$?

We have to find a vector that is orthogonal to two given vectors. And that is something that we've already learned: use cross products! $\vec{n}$ can be taken to be

$$\boxed{\vec{n} = \vec{v_1} \times \vec{v_2}}.$$

<u>Example</u> We can find an equation that contains three points using the above observation.

Let's say we want to find an equation for the plane that has $P = (2,3,1)$, $Q = (-1,0,2)$, $R = (3,2,0)$.



Then $\overrightarrow{PQ}, \overrightarrow{PR}$ are parallel to the plane.

$$\overrightarrow{PQ} = \langle -3, -3, 1 \rangle$$
$$\overrightarrow{PR} = \langle 1, -1, -1 \rangle$$

Thus a normal vector can be given by

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle (-3)\cdot(-1) - 1\cdot(-1),\ 1\cdot 1 - (-3)\cdot(-1),\ (-3)\cdot(-1) - (-3)\cdot 1 \rangle$$
$$= \langle 3+1, 1-3, 3+3 \rangle = \langle 4, -2, 6 \rangle.$$

Thus we have

$$4(x-2)-2(y-3)+6(z-1)=0, \text{ or}$$

$$4x-8-2y+6+6z-6=0, \text{ or}$$

$$4x-2y+6z=8.$$

Geometry of planes

Planes are the most important surface of all, so we will spend more on the geometry of planes.

We can think of the positioning of two planes, just like we did for lines.

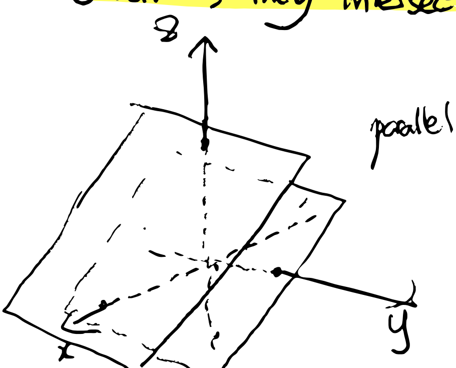
In this case, there is no "skew" position; either they are parallel or they intersect.

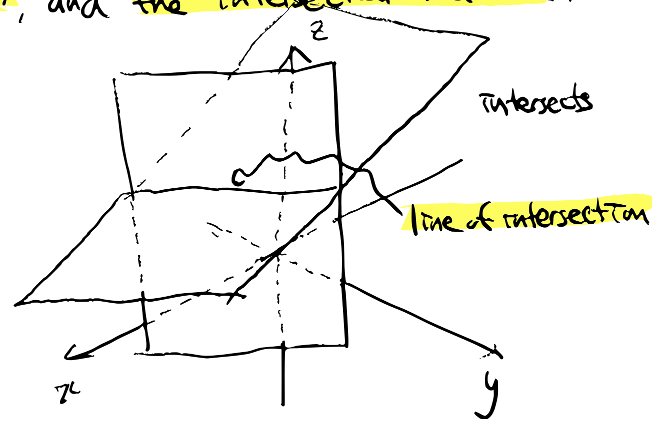
- They are parallel if the two normal vectors are parallel.

<u>Example</u> $2x+4y+z=3$ and $4x+8y+2z=1$ are parallel.

normal vector $\langle 2,4,1\rangle$ — normal vector $\langle 4,8,2\rangle$

- Otherwise, they intersect, and the intersection is a line.





Given two planes that intersect, how do we find the equation for the line of intersection?

Suppose we have two planes

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2.$$

We know that a directional vector of the line of intersection is orthogonal to both normal vectors $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$, since the line is contained in both planes! So a directional vector can be taken to be

$$\boxed{\langle a_1, b_1, c_1\rangle \times \langle a_2, b_2, c_2\rangle}$$

Now we only need to know a point that is contained in the line. To get this we solve an (underdetermined) system of linear equations $\begin{aligned} a_1x+b_1y+c_1z&=d_1,\\ a_2x+b_2y+c_2z&=d_2.\end{aligned}$

There are many solutions, but we only need to know one solution. This can be done more easily by setting one of x, y, z to be 0

Example Find the equation for the line of intersection of the two planes

$$x-y+3z=2$$
$$5x+2y+z=3.$$

Step 1 Find a directional vector.

This can be given by $\langle 1,-1,3\rangle \times \langle 5,2,1\rangle$.

$$\langle 1,-1,3\rangle \times \langle 5,2,1\rangle = \langle (-1)\cdot 1 - 3\cdot 2,\ 3\cdot 5 - 1\cdot 1,\ 1\cdot 2 - (-1)\cdot 5\rangle$$

$$= \langle -1-6, 15-1, 2+5\rangle$$

$$= \langle -7, 14, 7\rangle.$$

Step 2 Find a point on it.

We solve $\quad x - y + 3z = 2$

$\quad 5x + 2y + z = 3.$

Following the advice, we let $x=0$.

Then we're solving $\quad -y+3z=2 \cdots$ A

$\quad 2y+z=3 \cdots$ B

2A is $-2y+6z=4$, so $\quad$ 2A + B $\rightsquigarrow$ $7z=7$, $z=1$.

Then $y=1$. So $(0,1,1)$ is on it.

Thus the line of intersection has a parametric equation

$$x = -7t$$
$$y = 1+14t$$
$$z = 1+7t.$$

More generally, we can talk about the angle between two planes, which is the acute angle between the normal vectors $\vec{n_1}, \vec{n_2}$

Namely $\boxed{\cos\theta = \left|\dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|.}$ ← this is the same as the case of the angle between two lines – θ never exceeds $\frac{\pi}{2}$.

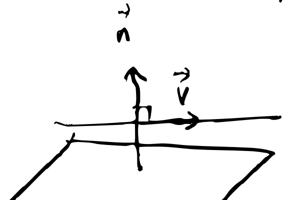
Lines and planes

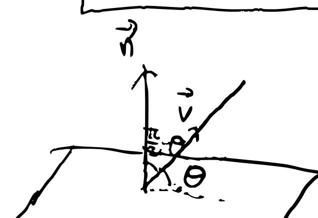
We can now talk about the relation between a line and a plane. Given a line and a plane (in 3D), they either are parallel or intersect.

- A line (point P + direction $\vec{v}$) is parallel to a plane (point Q + normal vector $\vec{n}$) if $\vec{n}$ is orthogonal to $\vec{v}$, namely $\vec{n}\cdot\vec{v}=0$.

  More generally, the angle between a line and a plane is θ, where $\frac{\pi}{2}-\theta$ is the acute angle between the direction vector $\vec{v}$ and the normal vector $\vec{n}$. This means

  $$\cos\left(\frac{\pi}{2}-\theta\right) = \left|\frac{\vec{n}\cdot\vec{v}}{|\vec{n}||\vec{v}|}\right|, \text{ or } \boxed{\sin(\theta) = \left|\frac{\vec{n}\cdot\vec{v}}{|\vec{n}||\vec{v}|}\right|.}$$





- Otherwise, the line and the plane intersect at a point.

The point of intersection can be obtained by plugging the parametric equation for the line into the equation for the plane.

Example Find the angle between the line $\begin{aligned} x &= 1+3t \\ y &= -2t \\ z &= 1 \end{aligned}$

and the plane $3x-2y=16$. Find their point of intersection.

Solution We have

$$\sin\theta = \frac{\langle 3,-2,0\rangle \cdot \langle 3,-2,0\rangle}{\sqrt{3^2+(-2)^2+0^2}\,\sqrt{3^2+(-2)^2+0^2}} = \frac{13}{13} = 1, \text{ so } \theta = \frac{\pi}{2}.$$

To find the point of intersection, we plug $\begin{aligned} x &= 1+3t \\ y &= -2t \\ z &= 1 \end{aligned}$ into

$3x-2y=16$: $\quad 3(1+3t)-2(-2t)=16$, or

$3+13t=16$, or $t=1$. So the point of intersection

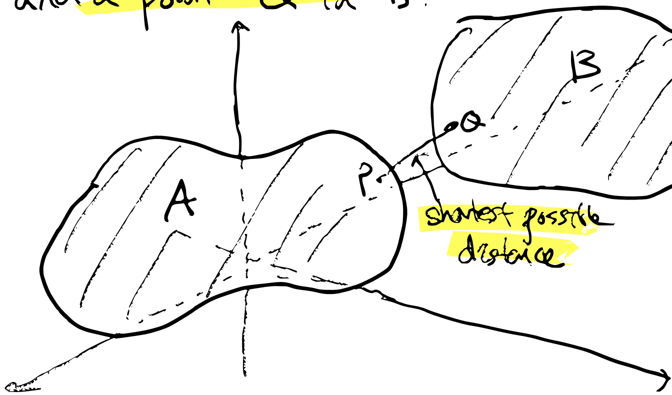
is $\begin{aligned} x &= 1+3=4 \\ y &= -2 \\ z &= 1. \end{aligned}$

Distance We know the distance between the two points

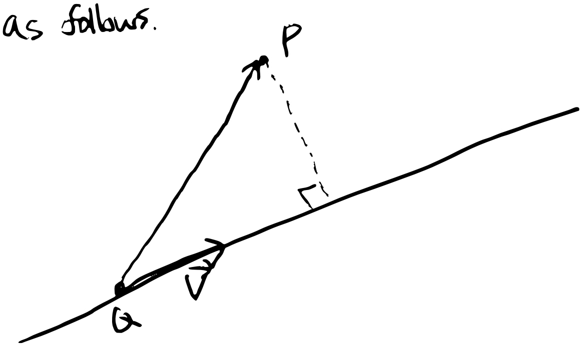
$P=(a,b,c)$ and $Q=(d,e,f)$ is $\sqrt{(a-d)^2+(b-e)^2+(c-f)^2}$.

We can talk about the distance between two objects that are not necessarily points.

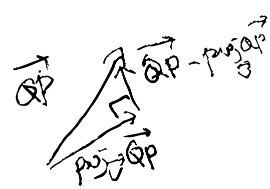
The distance between two objects A and B is the smallest possible value of the distance between a point P in A and a point Q in B.



<u>Example</u> The distance between a point P and a line (passing a point Q with direction vector $\vec{v}$) can be obtained as follows.



<u>Step 1</u>. $\overrightarrow{QP}$ can be decomposed as

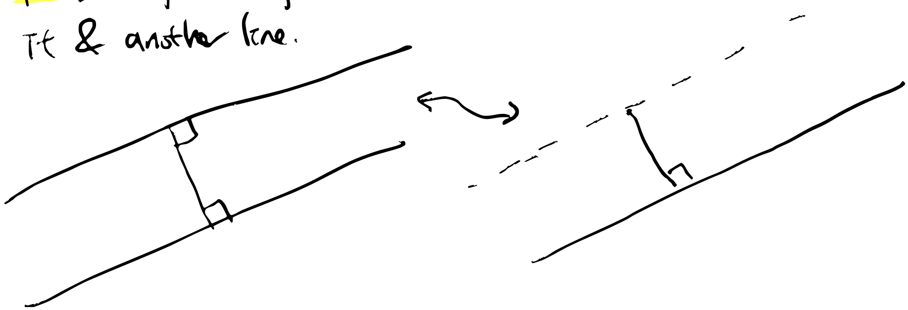


$$\overrightarrow{QP} = \underbrace{\text{proj}_{\vec{v}}\overrightarrow{QP}}_{\text{parallel to } \vec{v}} + \underbrace{(\overrightarrow{QP} - \text{proj}_{\vec{v}}\overrightarrow{QP})}_{\text{orthogonal to } \vec{v}.}$$

<u>Step 2</u> The distance is given by $\left|\overrightarrow{QP} - \text{proj}_{\vec{v}}\overrightarrow{QP}\right|$.

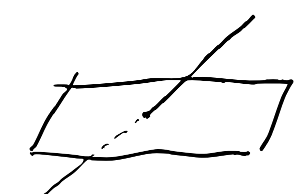
This makes sense, because

Using this, we find the distance between two parallel lines: pick a point from one line and find the distance between it & another line.

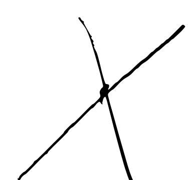


Example If two objects A, B intersect, the distance is 0.

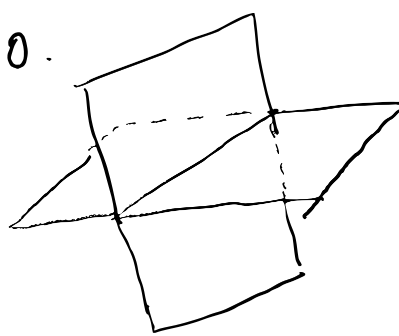
It's because you can take a point P that lies on both A and B, and the distance between P and itself is 0.



distance 0

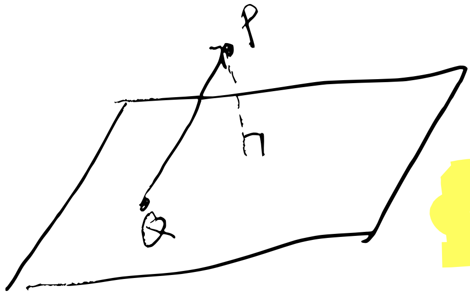


distance 0



distance 0.

Example What is the distance between a point P and a plane that passes through a point Q with normal vector $\vec{n}$?



If you see the left picture, it's pretty clear that it should be

$$|\mathrm{proj}_{\vec{n}} \overrightarrow{QP}|.$$

If you calculate, the distance between the point (s,t,u) and the plane $ax+by+cz+d=0$ is

$$\boxed{\frac{|as+bt+cu+d|}{\sqrt{a^2+b^2+c^2}}}.$$

Exercise Check it!

Using this, we can compute the distance between a line/plane and a plane that are parallel to each other

Finally, this can be used to find the distance between skew lines! Suppose we have two skew lines, L_1 passing through the point P_1 with directional vector $\vec{v_1}$, L_2 passing through the point P_2 with directional vector $\vec{v_2}$.

Then the plane A passing through P_1 parallel to both $\vec{v_1}$ and $\vec{v_2}$ contains L_1 and is parallel to L_2. So we can find the distance between A and L_2, which is the distance between L_1 and L_2.

Example Find the distance between

$$L_1 : x = 1+t, \quad y = -2+3t, \quad z = 4-t$$
$$L_2 : x = 2s, \quad y = 3+s, \quad z = -3+4s.$$

Solution First we figure out whether they are skew, parallel or intersecting.

We try to solve
$$1+t = 2s \quad \cdots \text{(A)}$$
$$-2+3t = 3+s \quad \cdots \text{(B)}$$
$$4-t = -3+4s \quad \cdots \text{(C)}$$

2(A) is $2+2t = 4s$, so 2(A) − (C) is $(2+2t)-(4-t) = 4s-(-3+4s)$
$= 3$,

or $3t-2=3$, or $3t=5$, or $t=\frac{5}{3}$.

So from (A) we have $\frac{8}{3} = 2s$, or $s = \frac{4}{3}$. We plug this to (B), and get $-2+5 = 3+\frac{4}{3}$, which is false.

So they don't intersect.

The direction vectors are $\vec{v_1} = \langle 1, 3, -1 \rangle$ for L_1

$\vec{v_2} = \langle 2, 1, 4 \rangle$ for L_2,

and they are not parallel, so the lines are skew.

We now consider the plane A that contains L_1 and parallels L_2.

Since L_1 passes through $(1, -2, 4)$ with directional vector

$\vec{v_1}$, A passes through $(1, -2, 4)$ and parallels $\vec{v_1}$ and $\vec{v_2}$,

or has a normal vector $\vec{n} = \vec{v_1} \times \vec{v_2}$.

So $\vec{n} = \langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle$

$$= \langle 3 \cdot 4 - (-1) \cdot 1,\ (-1) \cdot 2 - 1 \cdot 4,\ 1 \cdot 1 - 3 \cdot 2 \rangle$$

$$= \langle 12 + 1, -2 - 4, 1 - 6 \rangle = \langle 13, -6, -5 \rangle$$

So A is expressed as

$$13(x-1) - 6(y+2) - 5(z-4) = 0, \text{ or}$$

$$13x - 13 - 6y - 12 - 5z + 20 = 0, \text{ or}$$

$$13x - 6y - 5z - 5 = 0.$$

Since L_2 passes through $(0, 3, -3)$, the distance between L_1 and L_2 = distance between A and L_2 = distance between A and $(0, 3, -3)$. Now we can use the formula

$$\text{Distance} = \boxed{\frac{|as+bt+cu+d|}{\sqrt{a^2+b^2+c^2}}}$$

$$= \frac{|13\cdot 0 - 6\cdot 3 - 5\cdot(-3) - 5|}{\sqrt{13^2+(-6)^2+(-5)^2}}$$

$$= \frac{|-18+15-5|}{\sqrt{169+36+25}} = \boxed{\frac{8}{\sqrt{230}}}$$